SIMULTANEOUS APPROXIMATION AND INTERPOLATION OF FUNCTIONS ON CONTINUA IN THE COMPLEX PLANE

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Abstract

We construct polynomial approximations of Dzjadyk type (in terms of the k-th modulus of continuity, $k \geq 1$) for analytic functions defined on a continuum E in the complex plane, which simultaneously interpolate at given points of E. Furthermore, the error in this approximation is decaying as $e^{-cn^{\alpha}}$ strictly inside E, where c and α are positive constants independent of the degree n of the approximating polynomial.

Key words: polynomial approximation, interpolation, analytic functions, quasiconformal curve.

AMS subject classification: 30E10, 41A10

1. Introduction and main results

Let $E \subset \mathbf{C}$ be a compact set with connected complement $\Omega := \overline{\mathbf{C}} \setminus E$, where $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ is the extended complex plane. Denote by $A(E)$ the class of all functions continuous on E and analytic in E^0, the interior of E (the case $E^0 = \emptyset$ is not excluded). Let $\mathbf{P}_n$, $n \in \mathbf{N}_0 := \{0, 1, 2, \ldots\}$, be the class of complex polynomials of degree at most n. For $f \in A(E)$ and $n \in \mathbf{N}_0$, define

$$E_n(f, E) := \inf_{p \in \mathbf{P}_n} ||f - p||_E ,$$

where $|| \cdot ||_E$ denotes the uniform norm on E. By Mergelyan's theorem (see [13]), we have that

$$\lim_{n\to\infty} E_n(f, E) = 0 \quad (f \in A(E)).$$

The following assertion on "simultaneous approximation and interpolation" quantifies a result of Walsh [38, p. 310]: Let $z_1, \ldots, z_N \in E$ be distinct points,

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$f \in A(E)$. Then for any $n \in \mathbf{N} := \{1, 2, \ldots\}$, $n \geq N-1$, there exists a polynomial $p_n \in \mathbf{P}_n$ such that

$$(1.1) \qquad ||f - p_n||_E \leq c\, E_n(f, E),$$
$$p_n(z_j) = f(z_j) \quad (j = 1, \ldots, N),$$

where $c > 0$ is independent of n and f.

A suitable polynomial has the form

$$p_n(z) = p_n^*(z) + \sum_{j=1}^{N} \frac{q(z)}{q'(z_j)(z - z_j)}(f(z_j) - p_n^*(z_j)),$$

where

$$q(z) := \prod_{j=1}^{N}(z - z_j),$$

and $p_n^* \in \mathbf{P}_n$ satisfies

$$||f - p_n^*||_E = E_n(f, E).$$

It is natural to ask whether it is possible to interpolate the function f as before at arbitrary prescribed points and to simultaneously approximate it in an even stronger sense than in (1.1). The theorem of Gopengauz [18] about simultaneous polynomial approximation of real functions continuous on the interval $[-1, 1]$ and their interpolation at ± 1 is an example of such result. For recent accounts of improvements and generalizations of this remarkable statement (for real functions) we refer the reader to [23], [35] and [19].

We shall make use of the D-approximation (named after Dzjadyk, who found in the late 50's - early 60's a constructive description of Hölder classes requiring a nonuniform scale of approximation) as a substitute for (1.1). There is an extensive bibliography devoted to this subject (see, for example, the monographs [13], [36], [17], [28] and [7]). In the overwhelming majority of the results on D-approximation, E is a continuum (one of the rare exceptions is the recent interesting paper [30]). In [3] it is shown that, for the D-approximation to hold for a continuum E, it is sufficient and under some mild restrictions also necessary that E belongs to the class H^*, which can be defined as follows (cf. [2] and [5]).

From now on we assume that E is a continuum with $\operatorname{diam} E > 0$, connected complement Ω and boundary $L := \partial E$. In the sequel, we denote by α, β, $c, c_1, \ldots$ positive constants (possibly different at different occurences) that either are absolute or depend on parameters not essential for the arguments; otherwise, such a dependence will be indicated.

We say that $E \in H$ if any points $z, \zeta \in E$ can be joined by an arc $\gamma(z,\zeta) \subset E$ whose length $|\gamma(z,\zeta)|$ satisfies the condition

$$(1.2) \qquad |\gamma(z,\zeta)| \le c\,|z-\zeta|, \quad c = c(E) \ge 1.$$

Let us compactify the domain Ω by prime ends in the Caratheodory sense (see [22]). Let $\tilde{\Omega}$ be this compactification, and let $\tilde{L} := \tilde{\Omega} \backslash \Omega$. Assuming that $E \in H$, then all the prime ends $Z \in \tilde{L}$ are of the first kind, i.e., they have singleton impressions $|Z| = z \in L$. The circle $\{\xi : |\xi - z| = r\}$, $0 < r < \frac{1}{2}\operatorname{diam} E$, contains one arc, or finitely many arcs, dividing Ω into two subdomains: an unbounded subdomain and a bounded subdomain such that Z can be defined by a chain of cross-cuts of the bounded subdomain. Let $\gamma_Z(r)$ denote that one of these arcs for which the unbounded subdomain is as large as possible (for given Z and r). Thus, the arc $\gamma_Z(r)$ separates the prime end Z from ∞ (cf. [8], [7]).

If $0 < r < R < \frac{1}{2}\operatorname{diam} E$, then $\gamma_Z(r)$ and $\gamma_Z(R)$ are the sides of some quadrilateral $Q_Z(r,R) \subset \Omega$ whose other two sides are parts of the boundary L. Let $m_Z(r,R)$ be the module of this quadrilateral, i.e., the module of the family of arcs that separate the sides $\gamma_Z(r)$ and $\gamma_Z(R)$ in $Q_Z(r,R)$ (see [1], [20]).

We say that $E \in H^*$ if $E \in H$ and if there exist constants $c = c(E) < \frac{1}{2}\operatorname{diam} E$ and $c_1 = c_1(E)$ such that

$$(1.3) \qquad |m_Z(|z-\zeta|, c) - m_{\mathcal{Z}}(|z-\zeta|, c)| \le c_1$$

for any pair of prime ends $Z, \mathcal{Z} \in \tilde{L}$, with their impressions $z = |Z|$, $\zeta = |\mathcal{Z}|$ satisfying $|z-\zeta| < c$.

In particular, H^* includes domains with quasiconformal boundaries (see [1], [20]) and the classes B_k^* of domains introduced by Dzjadyk [13]. For a more detailed investigation of the geometric meaning of conditions (1.2) and (1.3), see [5].

We will be studying functions defined by their k-th modulus of continuity $(k \in \mathbf{N})$. There is a number of different definitions of these moduli in the complex plane (see [37], [36], [11], [27]). The definition by Dyn'kin [11] is the most convenient for our purpose here.

From now on, suppose that $E \in H^*$. Set

$$D(z,\delta) := \{\zeta : |\zeta - z| \le \delta\} \quad (z \in \mathbf{C},\ \delta > 0).$$

The quantity

$$\omega_{f,k,z,E}(\delta) := E_{k-1}(f, E \cap D(z,\delta)),$$

where $f \in A(E)$, $k \in \mathbf{N}$, $z \in E$, $\delta > 0$, is called the k-*th local modulus of continuity*, and

$$\omega_{f,k,E}(\delta) := \sup_{z\in E} \omega_{f,k,z,E}(\delta)$$

is called the k-*th (global) modulus of continuity* of f on E. It is known (see [36]) that the behavior of this modulus is essentially the same as in the classical case of the interval $E = [-1, 1]$. In particular,

$$\omega_{f,k,E}(t\delta) \le c\, t^k\, \omega_{f,k,E}(\delta) \quad (t > 1, \delta > 0). \tag{1.4}$$

We denote by $A^r(E)$, $r \in \mathbf{N}$, the class of functions $f \in A(E)$ which are r-times continuously differentiable on E, where we set $A^0(E) := A(E)$.

By definition, the function $w = \Phi(z)$ maps Ω conformally and univalently onto $\Delta := \{w : |w| > 1\}$ and is normalized by the conditions

$$\Phi(\infty) = \infty, \ \Phi'(\infty) > 0. \tag{1.5}$$

The same symbol Φ denotes the homeomorphism between the compactification $\tilde{\Omega}$ of Ω and $\overline{\Delta}$, which coincides with $\Phi(z)$ in Ω. Let $\Psi := \Phi^{-1}$. We define the distance to the level curves of $\Phi(z)$

$$L_\delta := \{\zeta : |\Phi(\zeta)| = 1 + \delta\} \quad (\delta > 0)$$

by

$$\rho_\delta(z) := \operatorname{dist}(z, L_\delta) \quad (z \in \mathbf{C},\, \delta > 0),$$

where

$$\operatorname{dist}(\zeta, B) := \inf\{|\zeta - z| : z \in B\} \quad (\zeta \in \mathbf{C},\, B \subset \mathbf{C}).$$

Theorem 1 *Let $E \in H^*$, $f \in A(E)$, $k \in \mathbf{N}$, and let $z_1, \ldots, z_N \in E$ be distinct points. Then for any $n \in \mathbf{N}$, $n \ge N + k$, there exists a polynomial $p_n \in \mathbf{P}_n$ such that*

$$|f(z) - p_n(z)| \le c_1\, \omega_{f,k,E}(\rho_{1/n}(z)) \quad (z \in L), \tag{1.6}$$

$$p_n(z_j) = f(z_j) \quad (j = 1, \ldots, N) \tag{1.7}$$

with c_1 independent of n.

Moreover, if $E^0 \neq \emptyset$ and if for any $0 < \delta < 1$, there is a constant c_2 such that

$$\int_0^\delta \omega_{f,k,E}(t)\, \frac{dt}{t} \le c_2\, \omega_{f,k,E}(\delta), \tag{1.8}$$

then, in addition to (1.6) and (1.7),

$$||f - p_n||_K \le c_3 \exp(-c_4 n^\alpha) \tag{1.9}$$

for every compact set $K \subset E^0$ *, where the constants* c_3, c_4 *and* $0 < \alpha \le 1$ *are independent of* n *.*

A polynomial p_n satisfying (1.6) is called a D-approximation of the function f (D-property of E, Dzjadyk type direct theorem). For $k > 1$, (1.6) generalizes the corresponding direct theorems of Belyi and Tamrazov [9] (when E is a quasidisk) and Shevchuk [27] (when E belongs to the Dzjadyk class B_k^*). More detailed history can be found in these papers.

It was first noticed by Shirokov [29] that the rate of D-approximation may admit significant improvement strictly inside E. Saff and Totik [25] proved that if L is an analytic curve, then an exponential rate is achievable strictly inside E, while on the boundary the approximation is "near-best". However, even for domains with piecewise smooth boundary without cusps (and therefore belonging to H^*), the error of approximation strictly inside E cannot be better than e^{-cn^α} (cf. (1.9)), where α may be arbitrarily small (see [21], [32]). In the results from [21], [32], [31] containing estimates of the form (1.9), it is usually assumed that Ω satisfies a wedge condition. For a continuum $E \in H^*$, this condition can be violated.

Keeping in mind the Gopengauz result [18], we generalize Theorem 1 to the case of the Hermite interpolation and simultaneous approximation of a function $f \in A^r(E)$ and its derivatives. For simplicity we formulate and prove this assertion only for the case of boundary interpolation points and without the analog of (1.9).

Theorem 2 *Let* $E \in H^*$, $f \in A^r(E)$, $r \in \mathbf{N}$, $k \in \mathbf{N}$ *, and let* $z_1, \dots, z_N \in \partial E$ *be distinct points. Then for any* $n \in \mathbf{N}$, $n \ge Nr + k$ *, there exists a polynomial* $p_n \in \mathbf{P}_n$ *such that for* $l = 0, \dots, r$ *,*

$$|f^{(l)}(z) - p_n^{(l)}(z)| \le c\, \rho_{1/n}^{r-l}(z)\, \omega_{f^{(r)},k,E}(\rho_{1/n}(z)) \quad (z \in L), \tag{1.10}$$

and

$$p_n^{(l)}(z_j) = f^{(l)}(z_j) \quad (j = 1, \dots, N), \tag{1.11}$$

with c *independent of* n *.*

Our next goal is to allow the number of interpolation nodes N to grow infinitely with the degree of approximating polynomial n. It is well known that we cannot take $N - 1$ equal to n, preserving uniform convergence (cf. Faber's theorem

[16] claiming that for $E = [-1, 1]$ there is no universal set of nodes such that the Lagrange interpolating polynomials converge to every continuous function in uniform norm). However, it was first observed by Bernstein [10] that for any continuous function on $E = [-1, 1]$ and any small $\varepsilon > 0$, there exists a sequence of polynomials interpolating in the Chebyshev nodes and uniformly convergent on $[-1, 1]$, such that $n \le (1+\varepsilon)N$. This result was developed in several directions. In particular, Erdős (see [14] and [15]) found a necessary and sufficient condition on the system of nodes, for this type of simultaneous approximation and interpolation to be valid.

We generalize the results of Bernstein and Erdős in the following Theorem. In order to accomplish this, we specify the choice of points $z_1, \ldots, z_N$ in an optimal fashion from the point of view of interpolation theory. Namely, we require that the discrete measure

$$\mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{z_j},$$

where δ_z denotes the unit mass placed at z, is close to the equilibrium measure for E (for details, see [26]). Fekete points (see [22], [26]) are natural candidates for this purpose.

A Jordan curve is called quasiconformal if it is an image of the unit circle under a quasiconformal homeomorphism of the complex plane onto itself, with infinity as a fixed point (see [20] for details).

Theorem 3 *Let E be a closed Jordan domain bounded by a quasiconformal curve L. Let f, r, k be as in Theorem 1 and let $z_1, \ldots, z_N \in E$ be the points of an N-th Fekete point set of E. Then for any $\varepsilon > 0$ there exists a polynomial $p_n \in \mathbf{P}_n$, $n \le (1+\varepsilon)N$, satisfying conditions (1.6) and (1.7). Moreover, if (1.8) holds then in addition to (1.6) and (1.7) we have (1.9), and the constants c_1, c_3, c_4 and α are independent of N.*

2. Auxiliary results

In this section, we give some results from [2]-[5], [8], which are needed for the proofs of the above theorems and which characterize the properties of the mappings Φ and Ψ in the case $E \in H^*$. For $a > 0$ and $b > 0$, we will use the expression $a \preceq b$ (order inequality) if $a \le cb$. The expression $a \asymp b$ means that $a \preceq b$ and $b \preceq a$ simultaneously. The distance $\rho_\delta(z)$ to the level lines of Φ is, for any $z \in L$, a normal majorant (in the terminology of [36]), i.e.,

(2.1) $$\rho_{2\delta}(z) \preceq \rho_\delta(z) \quad (\delta > 0).$$

Let $z, \zeta \in L,\ \delta > 0$. The condition $|z-\zeta| \preceq \rho_\delta(z)$ yields

$$\rho_\delta(\zeta) \asymp \rho_\delta(z). \tag{2.2}$$

If L is a quasiconformal curve, $z \in L,\ \zeta \in \Omega$ and if $|z-\zeta| \geq \rho_\delta(z)$, then the inequality

$$\frac{\rho_\delta(z)}{|z-\zeta|} \preceq \left(\frac{\delta}{|\Phi(z)-\Phi(\zeta)|} \right)^{\alpha} \tag{2.3}$$

holds with some $\alpha = \alpha(E)$.

One of the fundamental problems that, as a rule, is encountered in the construction of approximations by polynomials, is the problem of approximating the Cauchy kernel $1/(\zeta - z),\ z \in E,\ \zeta \in \overline{\Omega}$, by polynomial kernels of the form

$$K_n(\zeta, z) = \sum_{j=0}^{n} a_j(\zeta)\, z^j. \tag{2.4}$$

The most general kernels of such type, the functions $K_{r,m,k,n}(\zeta, z)$, were introduced by Dzjadyk (see [13, Chapter 9] or [7, Chapter 3]). Taking them as a basis for our discussion, we can establish the following result (cf. [3, Lemma 9]).

Lemma 1 *Let $E \in H^*$, and let $m,\ r \in \mathbf{N}$. Then for any $n \in \mathbf{N}$ there exists a polynomial kernel of the form (2.4) such that the following relations hold for $l = 0, \dots, r$, $z \in L$ and $\zeta \in \overline{\Omega}$ with $d(\zeta, E) \leq 3$:*

$$\left| \frac{\partial^l}{\partial z^l} \left(\frac{1}{\zeta - z} - K_n(\zeta, z) \right) \right| \leq \frac{c_1}{|\zeta - z|^{l+1}} \left(\frac{\rho_{1/n}(z)}{|\zeta - z| + \rho_{1/n}(z)} \right)^m,$$

$$\left| \frac{\partial^l}{\partial z^l} K_n(\zeta, z) \right| \leq \frac{c_2}{(|\zeta - z| + \rho_{1/n}(z))^{l+1}}, \tag{2.5}$$

where $c_j = c_j(m, r, E),\ j = 1, 2$.

In order to improve the approximation properties of the polynomial kernel $K_n(\zeta, z)$ inside of E, we use an idea from [31, Theorem 2], completing it by the following geometrical fact. Let

$$d(\zeta, B) := \operatorname{dist}(\zeta, B) = \inf\{|\zeta - z| : z \in B\} \quad (\zeta \in \mathbf{C},\ B \subset \mathbf{C}).$$

Lemma 2 *Let $E \in H^*,\ E^0 \neq \emptyset$. For any $\zeta \in \overline{\Omega}$ with $d(\zeta, L) \leq 3$, there exists a Jordan domain G_ζ with the following properties:*

(i) $\zeta \in \partial G_\zeta,\ E \subset \overline{G_\zeta}$;

(ii) $\operatorname{diam} G_\zeta \le c$;

(iii) ∂G_ζ *is* K*-quasiconformal.*

Here, the constants $c > \operatorname{diam} E$ *and* $K \ge 1$ *are independent of* ζ.

Proof. If $\zeta \in \Omega$ we set $\mathcal{Z} := \zeta$; if $\zeta \in L$ we denote by $\mathcal{Z} \in \tilde{L}$ the prime end whose impression coincides with ζ (or any of such prime ends). Let

$$\Gamma_\zeta := \{\xi \in \Omega : \arg \Phi(\xi) = \arg \Phi(\mathcal{Z})\}.$$

By virtue of [4, Lemma 1 and Lemma 2],

$$d(z, L) \succeq |z - \zeta| \quad (z \in \Gamma_\zeta), \tag{2.6}$$

and for any $z_1, z_2 \in \Gamma_\zeta$ the length of the part of Γ_ζ between these points satisfies

$$|\Gamma_\zeta(z_1, z_2)| \preceq |z_1 - z_2|. \tag{2.7}$$

A result of Rickman [24] (see also [7, p. 144]) together with (2.7) imply that Γ_ζ is K_1-quasiconformal with some $K_1 \ge 1$ independent of ζ, i.e., there exists a K_1-quasiconformal mapping $F : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that

$$F(\zeta) = 0, \ F(\infty) = \infty, \ F(\Gamma_\zeta) = \{w : w > 0\}.$$

We can assume that $|F(z_0)| = 1$ for a fixed $z_0 \in E^0$. We recall the following well-known property of quasiconformal automorphisms of the complex plane (see, for example, [7, p. 98]): If $|\xi_1 - \xi_2| \preceq |\xi_1 - \xi_3|$ then

$$|F(\xi_1) - F(\xi_2)| \preceq |F(\xi_1) - F(\xi_3)| \tag{2.8}$$

and vice versa.

According to (2.6) and (2.8) there are constants c_1 and c_2 such that

$$F(E) \subset G'_\zeta := \{w = re^{i\theta} : 0 \le r < c_1, \, c_2 < |\theta| \le \pi\}.$$

By the Ahlfors criterion (see [1], [20, p. 100]), $\partial G'_\zeta$ is K_2-quasiconformal with $K_2 = K_2(c_1, c_2) \ge 1$. Therefore, by (2.8) the domain $G_\zeta := F^{-1}(G'_\zeta)$ satisfies the conditions (i)-(iii) with $K = K_1 K_2$.

□

Let E, ζ and G_ζ be as in Lemma 2 and let $z_0 \in E^0$ be fixed. Consider the conformal mapping $\Phi_\zeta : \overline{\mathbf{C}} \setminus \overline{G_\zeta} \to \Delta$ normalized as in (1.5), and the conformal mapping $\phi_\zeta : G_\zeta \to \{w : |w - \frac{1}{2}| < \frac{1}{2}\}$ normalized by the conditions

$$\phi_\zeta(z_0) = \frac{1}{2}, \ \phi_\zeta(\zeta) = 1.$$

Next, we use results from the theory of local distortion, under conformal mappings of an arbitrary simply connected domain onto a canonical one, developed by Belyi [8] (see also [7]).

Lemma 2 as well as [8, Theorem 1 and Theorem 6] imply that the functions Φ_ζ^{-1} and ϕ_ζ satisfy a Hölder condition (with constants independent of ζ). Therefore, by [8, Theorem 4] for any $M \in \mathbf{N}$ there exists a polynomial $t_M(\zeta, z) \in \mathbf{P}_M$ (in z) such that

$$||\phi_\zeta - t_M(\zeta, \cdot)||_{\overline{G_\zeta}} \leq \frac{c_1}{M^\beta}$$

with some c_1 and β independent of ζ. We can assume that $t_M(\zeta, \zeta) = 1$.

Now for $n \in \mathbf{N}$, we set

$$M := \left[\frac{n^{1/(1+\beta)}}{2}\right], \ N := [n^{\beta/(1+\beta)}]$$

(here $[x]$ denotes the Gauss bracket of x, the largest integer not exceeding x) and we note that, for the polynomial

$$u_{n/2}(\zeta, z) := t_M^N(\zeta, z),$$

the inequality

$$||u_{n/2}(\zeta, \cdot)||_E \leq \left(1 + \frac{c_1}{M^\beta}\right)^N \preceq 1 \tag{2.9}$$

holds, as well as for any compact set $K \subset E^0$ and $\alpha := \beta/(1+\beta)$,

$$||u_{n/2}(\zeta, \cdot)||_K \leq (1 - c_2)^N \leq e^{-cn^\alpha}, \tag{2.10}$$

where the constants $c_2 < 1$ and c are independent of ζ.

Hence, the function defined by

$$T_n(\zeta, z) := \frac{1 - u_{n/2}(\zeta, z)}{\zeta - z} + u_{n/2}(\zeta, z)\, K_{[n/2]}(\zeta, z),$$

where $K_{[n/2]}(\zeta, z)$ is the polynomial kernel from Lemma 1, is a polynomial (in z) of degree at most n. According to Lemma 1, (2.9) and (2.10), it satisfies for $\zeta \in \overline{\Omega}, d(\zeta, L) \leq 3$, arbitrary but fixed $m \in \mathbf{N}$ and each compact set $K \subset E^0$ the following conditions:

$$\left|\frac{1}{\zeta - z} - T_n(\zeta, z)\right| = |u_{n/2}(\zeta, z)| \left|\frac{1}{\zeta - z} - K_{[n/2]}(\zeta, z)\right|$$

$$\preceq \begin{cases} \dfrac{1}{|\zeta - z|}\left(\dfrac{\rho_{1/n}(z)}{|\zeta - z| + \rho_{1/n}(z)}\right)^m, & \text{if } z \in L, \\ e^{-cn^\alpha}, & \text{if } z \in K. \end{cases} \tag{2.11}$$

In addition,

$$|T_n(\zeta,z)| \preceq \frac{1}{|\zeta-z|} \quad (z\in E,\, \zeta\in\overline{\Omega},\, d(\zeta,L)\le 3). \tag{2.12}$$

We will also need the continuous extension of an arbitrary function $F\in A(E)$ into the complex plane which preserves the smoothness properties of F. The corresponding construction, proposed by Dyn'kin [11], [12], is based on the Whitney partition of unity (see [34]) and local properties of the k-th modulus of continuity of F. A slight modification of the reasoning in [11], [12] and [34] gives the following result (cf. [7, pp. 13-15]).

Lemma 3 *Let $E\in H^*$. Any $F\in A(E)$ can be continuously extended to the complex plane (we preserve the notation F for the extension) such that:*

(i) $F(z)=0$ for z with $d(z,E)\ge 3$, i.e., F has compact support;

(ii) for $z\in \mathbf{C}\setminus E$,

$$\left|\frac{\partial F(z)}{\partial \overline{z}}\right| \le c_1\, \frac{\omega_{F,k,z^*,E}(23\, d(z,E))}{d(z,E)},$$

where $z^*\in E$ is an arbitrary point among those ones which are closest to z, $c_1=c_1(k,\operatorname{diam}E)$;

(iii) if $\zeta\in E,\, z\in\mathbf{C},\, |z-\zeta|<\delta,\, 0<\delta<\frac{1}{2}\operatorname{diam}E$, then

$$|F(z)-P_{F,k,\zeta,E,\delta}(z)| \le c_2\, \omega_{F,k,\zeta,E}(25\,\delta),$$

where $P_{F,k,\zeta,E,\delta}(z)\in\mathbf{P}_{k-1}$ is the (unique) polynomial such that

$$||F-P_{F,k,\zeta,E,\delta}||_{E\cap D(\zeta,\delta)} = \omega_{F,k,\zeta,E}(\delta),$$

and $c_2=c_2(k)$;

(iv) if F satisfies a Lipschitz condition on E, i.e.,

$$|F(z)-F(\zeta)|\le c\,|z-\zeta| \quad (z,\zeta\in E),$$

then the extension satisfies the same condition for $z,\zeta\in\mathbf{C}$, with $c_3=c_3(c,\operatorname{diam}E,k)$ instead of c.

3. Proof of Theorem 1

We fix a point $z_0 \in E$ and consider a primitive of f:

$$F(\zeta) := \int\limits_{\gamma(z_0,\zeta)} f(\xi)\, d\xi \quad (\zeta \in E), \tag{3.1}$$

where $\gamma(z_0, \zeta) \subset E$ is an arbitrary rectifiable arc joining z_0 and ζ.

On writing for $z \in L, \zeta \in E$ with $|\zeta - z| \le \delta$,

$$\begin{aligned} F(\zeta) &= F(z) + \int\limits_{\gamma(z,\zeta)} f(\xi)\, d\xi \\ &= \nu_\delta(\zeta, z) + \int\limits_{\gamma(z,\zeta)} (f(\xi) - P_{f,k,z,E,c\delta}(\xi))\, d\xi, \end{aligned}$$

where $c \ge 1$ is the constant from (1.2), we obtain

$$\omega_{F,k+1,z,E}(\delta) \le ||F - \nu_\delta(\cdot, z)||_{E\cap D(z,\delta)} \preceq \delta\, \omega(\delta),$$

where $\omega(\delta) := \omega_{f,k,E}(\delta)$. Using Lemma 3, we can extend F continuously to $\mathbf{C}$, so that F has compact support and satisfies

$$\left| \frac{\partial F(\zeta)}{\partial \overline{\zeta}} \right| \preceq \omega(d(\zeta, L)), \tag{3.2}$$

for $\zeta \in \Omega^* := \{\zeta \in \overline{\Omega} : d(\zeta, L) \le 3\}$. Moreover, for $z \in L, \zeta \in \mathbf{C}$ with $|z - \zeta| \le \delta < \frac{1}{2} \operatorname{diam} E$, we have

$$|F(\zeta) - \nu_\delta(\zeta, z)| \preceq \delta\, \omega(\delta). \tag{3.3}$$

Indeed, since for $\zeta \in E \cap D(z, \delta)$,

$$\begin{aligned} &|\nu_\delta(\zeta, z) - P_{F,k+1,z,E,\delta}(\zeta)| \\ \le\ &|F(\zeta) - \nu_\delta(\zeta, z)| + |F(\zeta) - P_{F,k+1,z,E,\delta}(\zeta)| \preceq \delta\, \omega(\delta), \end{aligned}$$

we have by the Bernstein-Walsh lemma [38, p. 77]

$$||\nu_\delta(\cdot, z) - P_{F,k+1,z,E,\delta}||_{D(z,\delta)} \preceq \delta\, \omega(\delta).$$

Hence (3.3) follows from the last inequality and assertion (iii) of Lemma 3.

Next, we consider the most complicated case, that is, $E^0 \neq \emptyset$ and (1.8) holds. We introduce the polynomial kernel $Q_{n/2}(\zeta, z) := T_{[n/2]}(\zeta, z)$, which by (2.11) and (2.12) satisfies

$$\left\| \frac{1}{\zeta - \cdot} - Q_{n/2}(\zeta, \cdot) \right\|_K \preceq e^{-cn^\alpha} \quad (\zeta \in \Omega^*) \tag{3.4}$$

on each compact set $K \subset E^0$, and

$$\left| \frac{1}{\zeta - z} - Q_{n/2}(\zeta, z) \right| \preceq \frac{1}{|\zeta - z|} \left(\frac{\rho_{1/n}(z)}{|\zeta - z| + \rho_{1/n}(z)} \right)^k \quad (z \in L), \tag{3.5}$$

$$|Q_{n/2}(\zeta, z)| \preceq \frac{1}{|\zeta - z|} \quad (z \in E). \tag{3.6}$$

Further, we consider the polynomial

$$t_n(z) = -\frac{1}{\pi} \int\limits_{\Omega^*} \frac{\partial F(\zeta)}{\partial \overline{\zeta}} \, Q_{n/2}^2(\zeta, z) \, dm(\zeta) \quad (z \in E),$$

where $dm(\zeta)$ means integration with respect to the two-dimensional Lebesgue measure (area). Let $z \in L$, $D := D(z, \rho)$, $\sigma := \partial D$, $\rho := \rho_{1/n}(z)$. According to assertion (iv) of Lemma 3, F is an ACL-function (absolutely continuous on lines parallel to the coordinate axes) in $\mathbf{C}$. Hence Green's formula can be applied here (see [20]) to obtain

$$\begin{aligned} f(z) - t_n(z) &= \frac{1}{\pi} \int\limits_{\Omega^* \setminus D} \frac{\partial F(\zeta)}{\partial \overline{\zeta}} \left(Q_{n/2}^2(\zeta, z) - \frac{1}{(\zeta - z)^2} \right) dm(\zeta) \\ &+ \frac{1}{\pi} \int\limits_{D} \frac{\partial F(\zeta)}{\partial \overline{\zeta}} \, Q_{n/2}^2(\zeta, z) \, dm(\zeta) \\ &+ f(z) - \frac{1}{2\pi i} \int\limits_{\sigma} \frac{F(\zeta)}{(\zeta - z)^2} \, d\zeta \\ &= U_1(z) + U_2(z) + U_3(z). \end{aligned} \tag{3.7}$$

The first two integrals in (3.7) can be estimated in an appropriate way by passing to polar coordinates and using (1.4), (1.8), (3.2), (3.5) as well as (3.6):

$$|U_1(z)| \preceq \int\limits_{\rho}^{c} \omega(t) \frac{\rho^{k+1}}{t^{k+2}} \, dt \preceq \omega(\rho) \, \rho \int\limits_{\rho}^{c} \frac{dt}{t^2} \preceq \omega(\rho), \tag{3.8}$$

$$(3.9) \qquad |U_2(z)| \preceq \int_0^{\rho} \frac{\omega(t)}{t} dt \preceq \omega(\rho).$$

In order to estimate the third term in (3.7), we note that

$$|f(z) - (\nu_\rho)'_\zeta(z,z)| = |f(z) - P_{f,k,z,E,c\rho}(z)| \le \omega(c\rho) \preceq \omega(\rho),$$

so that by (3.3):

$$(3.10) \quad |U_3(z)| \le |f(z) - (\nu_\rho)'_\zeta(z,z)| + \frac{1}{2\pi} \left| \int_\sigma \frac{F(\zeta) - \nu_\rho(\zeta,z)}{(\zeta - z)^2} d\zeta \right| \preceq \omega(\rho).$$

Comparing (3.7)-(3.10), we obtain that

$$(3.11) \qquad |f(z) - t_n(z)| \preceq \omega(\rho_{1/n}(z)) \quad (z \in L).$$

The estimate

$$(3.12) \qquad ||f - t_n||_K \le e^{-cn^\alpha},$$

for any compact set $K \subset E^0$, follows immediately from (3.2) and (3.4) by a straight-forward modification of the above reasoning.

To satisfy the interpolation condition (1.7), we argue as follows. Let $n > 2N$. We consider the polynomials

$$V_{n/2+1}(\zeta, z) := \begin{cases} 1 - (\zeta - z) Q_{n/2}(\zeta, z), & \text{if } \zeta \in L,\ z \in E, \\ 1, & \text{if } \zeta \in E^0,\ z \in E, \end{cases}$$

and

$$u_n(z) := \sum_{j=1}^{N} \frac{q(z)}{q'(z_j)(z - z_j)} \left(f(z_j) - t_n(z_j)\right) V_{n/2+1}(z_j, z).$$

By (3.4), (3.5), (3.11) and (3.12),

$$|u_n(z)| \preceq \begin{cases} \sum_j{}' \omega(\rho_{1/n}(z_j)) \left(\frac{\rho_{1/n}(z)}{|z - z_j| + \rho_{1/n}(z)} \right)^k, & \text{if } z \in L, \\ e^{-cn^\alpha}, & \text{if } z \in K, \end{cases}$$

where $\sum_j'$ means the sum in all j with $z_j \in L$. To show that

$$p_n(z) := t_n(z) + u_n(z)$$

satisfies (1.6), (1.7) and (1.9), it is sufficient to prove that the inequality

$$(3.13) \qquad \omega(\rho_{1/n}(\zeta)) \left(\frac{\rho_{1/n}(z)}{|z - \zeta| + \rho_{1/n}(z)} \right)^k \preceq \omega(\rho_{1/n}(z))$$

holds for any $z,\zeta\in L$.

This relation is trivial if $|\zeta-z|\le\rho_{1/n}(\zeta)$ (cf. (2.2)). Hence we may assume that $|\zeta-z|>\rho_{1/n}(\zeta)$. Then by (1.4),

$$\omega(\rho_{1/n}(\zeta))\left(\frac{\rho_{1/n}(z)}{|z-\zeta|+\rho_{1/n}(z)}\right)^k\le\omega(|\zeta-z|)\left(\frac{\rho_{1/n}(z)}{|\zeta-z|}\right)^k\preceq\omega(\rho_{1/n}(z)),$$

which completes the proof of (3.13).

Note that we used assumption (1.8) only for the estimation of $U_2(z)$ in (3.9). If we are interested only in relations (1.6) and (1.7), then we need to choose in the above reasoning $Q_{n/2}(\zeta,z)=K_{[n/2]}(\zeta,z)$, where $K_n(\zeta,z)$ is the polynomial kernel from Lemma 1. Then, instead of (3.9), we obtain by (2.5) that

$$|U_2(\zeta,z)|\preceq\int\limits_0^\rho\omega(t)\,\frac{t\,dt}{\rho^2}\preceq\omega(\rho),$$

and (1.8) becomes superfluous.

$\square$

4. Proof of Theorem 2

Since the scheme of this proof is the same as in the proof of Theorem 1, we describe it only briefly. We begin with the Taylor formula for a primitive F defined by (3.1):

$$F(\zeta)=F(z)+\sum_{j=1}^r\frac{f^{(j-1)}(z)}{j!}(\zeta-z)^j+\frac{1}{r!}\int\limits_{\gamma(z,\zeta)}(\zeta-\xi)^rf^{(r)}(\xi)d\xi,$$

where $z,\zeta\in E$ and an arc $\gamma(z,\zeta)\subset E$ joins these points and satisfies (1.2). Therefore, we have for $z\in L,\,\zeta\in E$ with $|z-\zeta|\le\delta$,

$$F(\zeta)=\kappa_\delta(\zeta,z)+\frac{1}{r!}\int\limits_{\gamma(z,\zeta)}(z-\xi)^r\left(f^{(r)}(\xi)-P_{f^{(r)},k,z,E,c\delta}(\xi)\right)d\xi,$$

where $c\ge1$ is the constant from (1.2) and $\kappa_\delta(\zeta,z)$ is a polynomial (in ζ) of degree $\le k+r$. Using Lemma 3, we extend F continuously, so that F has compact support and satisfies

$$\left|\frac{\partial F(\zeta)}{\partial\overline{\zeta}}\right|\preceq d(\zeta,L)^r\,\omega(d(\zeta,L))\quad(\zeta\in\Omega^*:=\{\zeta\in\overline{\Omega}:\,d(\zeta,L)\le3\}),$$

$$|F(\zeta)-\kappa_\delta(\zeta,z)| \preceq \delta^{r+1}\,\omega(\delta) \quad (z\in L,\ \zeta\in \mathbf{C},\ |\zeta-z|\le\delta),$$
where $\omega(\delta):=\omega_{f^{(r)},k,z,E}(\delta)$.

Next, we introduce the polynomial

$$t_n(z) = -\frac{1}{\pi}\int\limits_{\Omega^*}\frac{\partial F(\zeta)}{\partial\overline{\zeta}}\,\frac{\partial}{\partial z}K_n(\zeta,z)\,dm(\zeta) \quad (z\in E),$$

where $K_n(\zeta,z)$ is the polynomial kernel from Lemma 1 (with $m=2r$).

Let $l=0,\dots,r$ and let $z,\ D$ as well as σ be the same as in (3.7).

By Green's formula, we have that

$$\begin{aligned} f^{(l)}(z)-t_n^{(l)}(z) &= \frac{1}{\pi}\int\limits_{\Omega^*\setminus D}\frac{\partial F(\zeta)}{\partial\overline{\zeta}}\,\frac{\partial^{l+1}}{\partial z^{l+1}}\left(K_n(\zeta,z)-\frac{1}{\zeta-z}\right)dm(\zeta)\\ &+ \frac{1}{\pi}\int\limits_{D}\frac{\partial F(\zeta)}{\partial\overline{\zeta}}\,\frac{\partial^{l+1}}{\partial z^{l+1}}K_n(\zeta,z)\,dm(\zeta)\\ &+ f^{(l)}(z)-\frac{1}{2\pi i}\int\limits_{\sigma}F(\zeta)\frac{\partial^{l+1}}{\partial z^{l+1}}\frac{1}{\zeta-z}\,d\zeta. \end{aligned}$$

Reasoning as in the proof of (3.11), we obtain that

$$|f^{(l)}(z)-t_n^{(l)}(z)| \preceq \rho_{1/n}^{r-l}(z)\,\omega(\rho_{1/n}(z)) \quad (z\in L). \tag{4.1}$$

Further, we assume that $n>2N(r+1)$ and introduce the auxiliary polynomials

$$V_{n/2}(\zeta,z) := 1+\frac{(\zeta-z)^{r+1}}{r!}\,\frac{\partial^r}{\partial z^r}K_{[n/2]}(\zeta,z)$$

and

$$u_n(z) := \sum_{j=1}^{N}\frac{q^{r+1}(z)}{(z-z_j)^{r+1}}\,V_{n/2}(z_j,z)\,\sum_{s=0}^{r}A_{j,s}\,(z-z_j)^s,$$

where

$$A_{j,s} := \sum_{\nu=0}^{s}\frac{1}{\nu!(s-\nu)!}\left(f^{(\nu)}(z_j)-t_n^{(\nu)}(z_j)\right)\left(\frac{\partial^{s-\nu}}{\partial z^{s-\nu}}\frac{(z-z_j)^{r+1}}{q^{r+1}(z)}\right)\Bigg|_{z=z_j}.$$

According to the Hermite interpolation formula (see [33]), we have

$$u_n^{(l)}(z_j) = f^{(l)}(z_j)-t_n^{(l)}(z_j) \quad (j=1,\dots,N).$$

Therefore the polynomial

$$p_n := u_n + t_n$$

satisfies the interpolation condition (1.11).

Since

$$|A_{j,s}| \preceq \rho_{1/n}^{r-s}(z_j)\,\omega(\rho_{1/n}(z_j)),$$

we obtain by Lemma 1 for any $z \in L$,

$$\begin{aligned} |u_n(z)| &\preceq \sum_{j=1}^{N} \left(\frac{\rho_{1/n}(z)}{|z-z_j| + \rho_{1/n}(z)} \right)^{2r} \sum_{s=0}^{r} \rho_{1/n}^{r-s}(z_j)\,\omega(\rho_{1/n}(z_j))\,|z-z_j|^s \\ &\preceq \rho_{1/n}^{r}(z)\,\omega(\rho_{1/n}(z)), \end{aligned} \tag{4.2}$$

where we used (2.2) and the following inequality: for $z, \zeta \in L$ with $|\zeta - z| \geq \rho_{1/n}(z)$,

$$\left| \frac{\rho_{1/n}(z)}{z-\zeta} \right|^{2r} |z-\zeta|^r\,\omega(|z-\zeta|) \preceq \rho_{1/n}^{r}(z)\,\omega(\rho_{1/n}(z)).$$

By a theorem of Tamrazov [36] (see also [7, p. 187]), (4.2) yields

$$|u_n^{(l)}(z)| \preceq \rho_{1/n}^{r-l}(z)\,\omega(\rho_{1/n}(z)). \tag{4.3}$$

Combining (4.1) and (4.3), we obtain (1.10).

□

5. Proof of Theorem 3

We use the same scheme as in the proof of Theorem 1. Let (1.8) hold. We construct a polynomial $t_N \in \mathbf{P}_N$ such that

$$|f(z) - t_N(z)| \preceq \omega(\rho_{1/N}(z)) \quad (z \in L), \tag{5.1}$$

where $\omega(\delta) := \omega_{f,k,E}(\delta)$, and

$$||f - t_N||_K \leq e^{-cN^{\alpha}} \tag{5.2}$$

for any compact set $K \subset E^0$.

Let $m := [\varepsilon N]$. Consider the polynomial

$$u_{N+m}(z) := \sum_{j=1}^{N} \frac{q(z)}{q'(z_j)(z-z_j)} (f(z_j) - t_N(z_j)) V_{m+1}(z_j, z),$$

where

$$V_{m+1}(\zeta, z) := 1 - (\zeta - z)Q_m(\zeta, z) \quad (\zeta \in L,\, z \in E),$$

and $Q_m(\zeta, z) := T_m(\zeta, z)$ is a polynomial of degree at most m (in z) satisfying the inequalities (cf. (2.11))

$$\left|\frac{1}{\zeta - z} - Q_m(\zeta, z)\right| \preceq \frac{1}{|\zeta - z|}\left(\frac{\rho_{1/m}(z)}{|\zeta - z| + \rho_{1/m}(z)}\right)^{k+l} \quad (z, \zeta \in L) \tag{5.3}$$

(the choice of $l = l(E) > 0$ will be specified below) and

$$\left\|\frac{1}{\zeta - \cdot} - Q_m(\zeta, \cdot)\right\|_K \leq e^{-cm^\alpha} \quad (\zeta \in L) \tag{5.4}$$

on each compact set $K \subset E^0$.

Let $z \in L,\, \Phi(z) = e^{i\theta_0},\, \Phi(z_j) = e^{i\theta_j}$,

$$0 \leq \theta_1 < \theta_2 < \ldots < \theta_N < \theta_{N+1} := \theta_1 + 2\pi.$$

It is proved in [6] that

$$|\theta_{j+1} - \theta_j| \asymp \frac{1}{N} \quad (j = 1, \ldots, N). \tag{5.5}$$

We rename the points $\{e^{i\theta_j}\}_1^N$ by $\{e^{i\theta'_j}\}_1^\mu$, $\{e^{i\theta''_j}\}_1^\nu$ and $\{e^{i\theta'''_j}\}_1^{N-\mu-\nu}$ in such a way that

$$|\theta_0 - \theta'_j| \leq \frac{1}{m} \quad (j = 1, \ldots, \mu),$$

and $\theta_j = \theta''_j, \theta'''_j$ satisfy

$$|\theta_0 - \theta_j| > \frac{1}{m}, \quad (\theta_j \notin \{\theta'_1, \ldots, \theta'_\mu\}),$$

$$\theta_0 < \theta''_1 < \theta''_2 < \ldots < \theta''_\nu \leq \pi + \theta_0,$$

$$\theta_0 - \pi < \theta'''_{N-\mu-\nu} < \ldots < \theta'''_1 < \theta_0.$$

Equation (5.5) implies that

$$\mu \asymp \frac{1}{\varepsilon},\; \nu \asymp N - \mu - \nu \asymp N.$$

Furthermore, for the function

$$h(\theta, \theta_0) := (f(\Psi(e^{i\theta})) - t_N(\Psi(e^{i\theta})))V_{m+1}(\Psi(e^{i\theta}), \Psi(e^{i\theta_0}))$$

we have by (1.4), (2.2), (5.1) and (5.3),

$$|h(\theta'_j, \theta_0)| \preceq \omega(\rho), \tag{5.6}$$

$$(5.7)\qquad |h(\theta_j'',\theta_0)| \preceq \omega(|z-z_j''|)\left(\frac{\rho}{|z-z_j''|}\right)^{k+l} \preceq \omega(\rho)\left(\frac{\rho}{|z-z_j''|}\right)^{l},$$

$$(5.8)\qquad |h(\theta_j''',\theta_0)| \preceq \omega(|z-z_j'''|)\left(\frac{\rho}{|z-z_j'''|}\right)^{k+l} \preceq \omega(\rho)\left(\frac{\rho}{|z-z_j'''|}\right)^{l},$$

where $\rho := \rho_{1/m}(z)$, $z_j'' := \Psi(e^{i\theta_j''})$, $z_j''' := \Psi(e^{i\theta_j'''})$.

It follows from (5.2) and (5.4) that the polynomial

$$(5.9)\qquad p_{[(1+\varepsilon)N]}(z) := t_N(z) + u_{N+m}(z)$$

satisfies (1.7) and (1.9).

We choose l so that

$$\left|\frac{\rho}{\zeta - z}\right|^{l} \preceq \left(\frac{1}{m|\Phi(\zeta)-\Phi(z)|}\right)^{2},$$

for $\zeta \in L$ with $|\zeta - z| > \rho$ (cf. (2.3)).

Since

$$\begin{aligned} |u_{N+m}(z)| &\leq \sum_{j=1}^{\mu} |h(\theta_j',\theta_0)| + \sum_{j=1}^{\nu} |h(\theta_j'',\theta_0)| + \sum_{j=1}^{N-\mu-\nu} |h(\theta_j''',\theta_0)| \\ &\preceq \omega(\rho)\left(1 + \sum_{j=1}^{N}\frac{1}{j^2}\right) \preceq \omega(\rho), \quad z \in L, \end{aligned}$$

by (5.6)-(5.8), we obtain the desired inequality (1.6) by (2.1) and (5.1), for $p_{[(1+\varepsilon)N]}$ given by (5.9).

Taking in the above argument $Q_m(\zeta,z) := K_m(\zeta,z)$, we obtain equations (1.6) and (1.7) even without assumption (1.8).

□

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